Optimal policy of water extraction of coastal lagoon under state constraints

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Bionature

Outline

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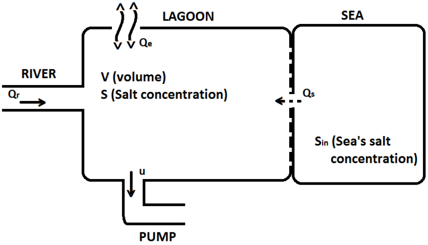
Tunquen lagoon

- Rare and fragile ecosystem due to intermittent connection to the sea
- Human exploitation: water-pumping by surrounding people

Question:

How much water can be pumped without damaging (too much) the ecosystem?

The model



Two phases: hot and wet

$$\alpha(t) := Q_e + Q_r + Q_s = \begin{cases} \alpha_h < 0 & \text{if } t \in [0, \overline{T}[, \\ \alpha_w > 0 & \text{if } t \in [\overline{T}, T], \end{cases}$$

and

$$Q_s(t) := \begin{cases} 0 & \text{if } t \in [0, \overline{T}[, \\ Q > 0 & \text{if } t \in [\overline{T}, T]. \end{cases}$$

The model

$$\begin{cases} \dot{V} = \alpha(t) - u, \\ \dot{S} = \frac{Q_s(t)S_{in} - \alpha(t)S}{V}, \end{cases} \quad \text{for a.e. } t \in [0, T].$$

The optimal control problem

The model:

$$\begin{cases} \dot{V} &= \alpha(t) - u, \\ \dot{S} &= \frac{Q_s(t)S_{in} - \alpha(t)S}{V}, \end{cases} \quad \text{for a.e. } t \in [0, T].$$

The optimal control problem:

Maximizing the accumulation of pumped water extracted on the time interval $[t_0, T]$:

$$\sup_{u \in \mathcal{U}} \int_{t_0}^{T} u(\tau) d\tau, \quad \text{where} \quad \mathcal{U} := \{u : [0, T] \to [0, 1] \text{ meas.}\},$$

with

$$\alpha_w < 1. \tag{H1}$$

The state constraint:

The solutions must stay between given bounds:

$$(V(t), S(t)) \in \mathcal{D} := [V_{min}, V_{max}] \times [S_{min}, S_{max}], \quad \forall t \in [t_0, T].$$

Outline

Definition and characterization with I.C. in the 2nd phase

Question:

Which initial conditions allow us to fulfill the state constraint restriction, i.e., that the solution remains in $\mathcal{D}$ at all time?

Definition

Given $t_0 \in [0, T]$, we define the (finite horizon) *viability kernel* of $\mathcal{D}$ on $[t_0, T]$ the set

$$\mathcal{A}(t_0) := \{(V_0, S_0) \in \mathcal{D}\ ;\ \exists u \in \mathcal{U} \text{ s.t. } (V_u^{t_0,V_0}(t), S_u^{t_0,V_0,S_0}(t)) \in \mathcal{D},\ \forall t \in [t_0, T]\}.$$

Let us assume

$$S^{eq} := \frac{QS_{in}}{\alpha_w} \in [S_{min}, S_{max}]. \tag{H2}$$

Proposition

If $t_0 \in [\overline{T}, T]$, then $\mathcal{A}(t_0) = \mathcal{D}$.

Characterization with I.C. in the 1st phase

Proposition

Let $t_0 \in [0, \overline{T})$, then

$$\mathcal{A}(t_0) = \text{hyp}\, \Lambda^{t_0} \cap \mathcal{D},$$

where Λ^{t_0} is the real valued function defined on $[V_{\min} - \alpha_h(\overline{T} - t_0), V_{\max}]$ by

$$\Lambda^{t_0}(V) := S_{\max} \frac{V + \alpha_h(\overline{T} - t_0)}{V}.$$

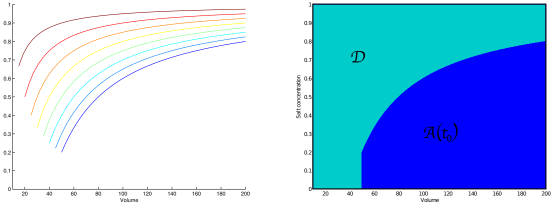


Figure: *Picture left*: The function $V \in [V_{\min} - \alpha_h(\overline{T} - t_0), V_{\max}] \mapsto \lambda^{t_0}(V)$ is ploted for $t_0 \in \{1, ..., 8\}$, $\bar{T} = 9$, $V_{min} = 10$, $V_{max} = 200$, $\alpha_h = -5$ and $S_{max} = 1$. *Picture right*: the set $\mathcal{A}(t_0)$ for $t_0 = 1$.

Outline

Optimal result on the second phase

We have $\dot{V} = \alpha(t) - u$ and we seek $\sup_{u\in\mathcal{U}} \int_{t_0}^{T} u(\tau)d\tau$, therefore we are interested in

$$\inf_{u\in\mathcal{U}} V_u^{t_0,V_0}(T).$$

Proposition

Whenever $(t_0, V_0, S_0) \in [\overline{T}, T] \times \mathcal{D}$, a feed-back control for the optimal problem is defined on $[t_0, T] \times \mathcal{D}$ by

$$u[t, V, S] = \begin{cases} 1, \text{ if } V > V_{min}, \\ \alpha_w, \text{ if } V = V_{min}, \end{cases} \qquad \forall (t, V, S) \in [t_0, T] \times \mathcal{D}.$$

Optimal result on the first phase

Suppose $t_0 \in [0, \overline{T})$, then by the dynamic programming principle the problem reduces to

$$\inf_{u \in \mathcal{U}} V_u^{t_0, V_0}(\overline{T}).$$

Proposition

Let $t_0 \in [0, \overline{T})$ and define

$$\Upsilon^{t_0} : V \in [V_{\min} - (\alpha_h - 1)(\overline{T} - t_0), V_{\max}] \mapsto S_{\max} \left(\frac{V + (\alpha_h - 1)(\overline{T} - t_0)}{V} \right)^{\frac{\alpha_h}{\alpha_h - 1}}$$

and

$$\Gamma^{t_0} : V \in [V_{\min} - \alpha_h(\overline{T} - t_0), V_{\min} - (\alpha_h - 1)(\overline{T} - t_0)] \mapsto S_{\max} \frac{\nu^{t_0}(V)}{V} \left(\frac{V_{\min}}{\nu^{t_0}(V)} \right)^{\frac{\alpha_h}{\alpha_h - 1}}$$

with $\nu^{t_0}(V) := V_{\min} + (\alpha_h - 1)(V_{\min} - V - \alpha_h(\overline{T} - t_0))$.

1. If $(V_0, S_0) \in \text{hyp}\,\Gamma^{t_0}$, then $u^\star = 0 - 1$ and the switching time is known explicitly.
2. If $(V_0, S_0) \in \text{hyp}\,\Upsilon^{t_0}$, then $u^\star = 1$.
3. If $(V_0, S_0) \in \mathcal{A}(t_0) \setminus (\text{hyp}\,\Gamma^{t_0} \cup \text{hyp}\,\Upsilon^{t_0})$, then $u^\star = 0 - 1$ and the switching time is given by an implicit expression.

Optimal result on the first phase

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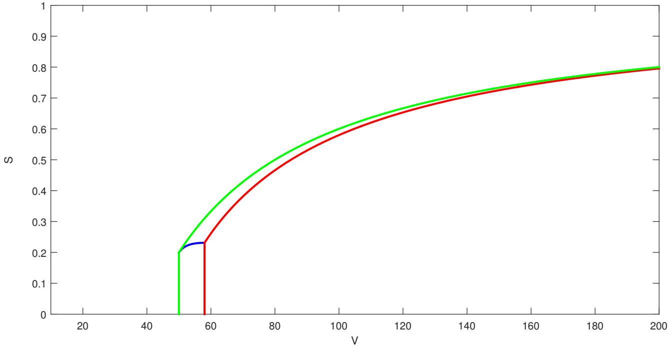

This is the end...

¡ gràcies per la vostra atenció !